



## Analysis Module

### [AN. 1]

Let  $\epsilon > 0$  be fixed. Show that the set of all real numbers  $x \in [0, 1]$  such that there exist infinitely many pairs  $p, q \in \mathbf{N}$  such that  $|x - p/q| < 1/q^2 + \epsilon$  has Lebesgue measure 0.

### [AN. 2]

Let  $f$  be a uniformly continuous function on  $\mathbf{R}$ . Suppose that  $f \in L^p$  for some  $p, 1 < p < \infty$ . Prove that  $f(x) \rightarrow 0$  as  $|x| \rightarrow \infty$ .

### [AN. 3]

(a) Give a definition of  $\|f\|_1$  of a measurable complex function  $f$ .

(b) Recall that the essential range of a function  $f \in L^1(\mathbf{R}; \mathbb{C})$  is the set consisting of complex numbers  $w$  such that

$$(\forall \epsilon > 0) \int_{\mathbf{R}} |f(x) - w| \chi_{\{|f(x) - w| < \epsilon\}} dx > 0$$

for every  $\epsilon > 0$ . Prove that  $R_f$  is compact.

(c) Show that  $\|f\|_1 = \sup_{w \in R_f} \int_{\mathbf{R}} |f(x) - w| dx$ .

### [AN. 4]

(a) Give a definition of a locally compact topological space.

(b) Give an example of a Borel measure  $\mu$  on  $\mathbf{R}$  such that  $X = L^2(\mathbf{R}; \mu)$  is locally compact and explain why it is so.

(c) Give an example of a Borel measure  $\mu$  on  $\mathbf{R}$  such that  $X = L^2(\mathbf{R}; \mu)$  is not locally compact and explain why it is so.

**Numerical Analysis module****[NA. 1] Quadrature and Newton's Method**

Let  $f(x) = \frac{1}{4}(x-5)^4 + x$ .

- (a) Compute  $f'(x); f''(x)$ . Is  $f$  convex? Explain your answer.
- (b) Find the minimizer of  $f(x)$ .
- (c) Write out the formula for Newton's method for function minimization.
- (d) Compute two Newton iterations, for  $x^0 = 4.5$ . Are the values approaching the minimum?
- (e) Approximate the integral  $\int_0^3 \frac{1}{x^2+2} dx$

[NA. 3] The conserved quantity  $q$  with flux function  $F$  satisfies the conservation law

$$(1) \quad \frac{\partial}{\partial t} q(x; t) + \frac{d}{dx} F(q; x; t) = 0; \quad \text{for } x \in [0; 1]$$

along with no-flux boundary conditions

$$F(q; x; t) = 0; \quad \text{for } x = 0; 1;$$

- (a) Show that the total mass of  $q$  is conserved.  
 (b) Assume that Fick's law of diffusion holds, so that  $F(q; x; t) = -D(x)q_x(x; t)$ . The energy is  $E(t) = \int_0^1 q^2(x; t) dx$ . Prove that the energy is non-increasing.  
 (c) Let  $G = [0; h; \dots; 1]$  be the finite difference grid, where  $h = 1/(n+1)$ . Let  $\partial_x^h$  be the forward difference operator on the grid. Let  $Q = (Q_0; \dots; Q_n)$  be a grid function. Write down the matrix  $M$  which maps the grid function  $Q$  to the grid function  $\partial_x^h Q$ , and includes the boundary conditions.  
 (d) Let  $Q(t) = (Q_0(t); \dots; Q_n(t))$  be a time-dependent grid function. Consider the method of lines for the PDE,

$$\frac{d}{dt} Q + M(\text{diag}(D))MQ$$

Prove that mass is conserved, and that the discrete energy  $E^h(t) = \frac{h}{2} \sum_{i=1}^n Q_i(t)^2$  is non-increasing.

[NA. 4]

- (a) Consider the initial value problem for the variable coefficient parabolic equation on the real line

$$u_t(x; t) + f(x; t)u_x(x; t) = g(x; t)$$

## Partial Differential Equations Module

[PDE 1.] We consider the boundary value problem

$$\begin{cases} \partial_y u + (2x + u) \partial_x u = x + 2u & \text{in } U \\ u(x; x) = 1 + x & \text{on } \partial U \end{cases} \quad (P)$$

where  $U = \{(x; y) : y > xg \text{ and } ; \geq \mathbb{R}\}$

- For which values of  $g$  and  $h$  does the problem (P) satisfy the noncharacteristic boundary condition?
- Give all solutions of the problem (P) in case  $g = 0$  and  $h = 1$ .
- Show that there does not exist any solution of the problem (P) in case  $g = 1$  and  $h \notin \mathbb{Z}$ .

[PDE 2.]

- Let  $U$  be an open and bounded subset of  $\mathbb{R}^n$ ,  $n \geq 1$ . Show that for any functions  $u, v \in C^2(U) \cap C^0(\bar{U})$  such that  $u \leq v$  in  $U$  and  $u = v$  on  $\partial U$ , we have  $u \leq v$  in  $U$ .
- Now we assume that  $n = 2$  and  $U = \{x \in \mathbb{R}^2 : R_1 < |x| < R_2\}$  for some real numbers  $R_2 > R_1 > 0$ . Show that for any function  $u \in C^2(U) \cap C^0(\bar{U})$  such that  $u = 0$  in  $U$ , we have

$$M(r) = \frac{M(R_1) \ln(R_2=r) + M(R_2) \ln(r=R_1)}{\ln(R_2=R_1)} \quad \forall r \in (R_1; R_2)$$

where  $M(r) = \sup_{|x|=r} u(x)$ .

Hint: Remember that the function  $v(x) = a + b \ln |x|$  is harmonic in  $\mathbb{R}^2 \setminus \{0\}$  for all  $a, b \in \mathbb{R}$ .

[PDE 3.] Let  $U$  be an open and bounded subset of  $\mathbb{R}^n$ ,  $n \geq 1$ , with smooth boundary. We consider the problem

$$\begin{cases} \Delta u = u & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases}$$